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J. Phys. A: Math. Gen. **33** (2000) 4203–4211. Printed in the UK PII: S0305-4470(00)12389-3

Quasi-exactly solvable quartic potentials with centrifugal and Coulombic terms

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Received 7 March 2000, in final form 5 April 2000

Abstract. A $\mathcal{PT}$-symmetrized radial Schrödinger equation in D dimensions is considered with complex potentials $V(x) = -x^4 + \mathrm{i}ax^3 + bx^2 + \mathrm{i}cx + \mathrm{i}dx^{-1}$. Finite (though arbitrarily large) multiplets of exact bound states are constructed analytically. Their elementary form is determined by a single, finite-dimensional secular equation. It interrelates their energies and couplings. In one dimension one gets Bender and Boettcher's $d = 0$ solutions.

1. Introduction

Bender and Boettcher (BB) [1] studied the three-parametric family of Hamiltonians defined on a certain complex curve $x = x(t) \in \mathbb{C}$, $t \in (-\infty, \infty)$ by the expression

$$
\begin{aligned}
&H^{(\mathrm{BB})} = -\partial_x^2 - x^4 + 2\mathrm{i}ax^3 + cx^2 + \mathrm{i}(a^3 - ac - 2J)x \\
&a, c \in \mathbb{R} \qquad J = 1, 2, \ldots .
\end{aligned} \tag{1}
$$

Their main result was an explicit construction of a J-plet of bound states in an elementary and closed exact form:

$$\psi_j(x) = \mathrm{e}^{-\mathrm{i}x^3/3 - ax^2/2 - \mathrm{i}bx} P_{j,J-1}(x) \qquad b = (a^2 - c)/2 \qquad j = 1, 2, \ldots, J$$

with $P_{j,J-1}(x)$ denoting certain polynomials of $(J-1)$th degree. The normalizability of the multiplet was a simple consequence of the requirement $x(t) \sim (\cos\varphi \mp \mathrm{i}\sin\varphi)t$ for asymptotics $t \to \pm\infty$ with $\varphi \in (0, \pi/3)$. One may even stay on the straight line $x(t) = t - \mathrm{i}\varepsilon$, provided only that $\mathrm{Re}\,(2\varepsilon + a) > 0$.

In an older paper Buslaev and Grecchi (BG) [2] proved that Hamiltonians (1) have a real and discrete spectrum. Their paper shows that these complex models find an immediate application in physics: their spectra coincide with the energy levels in a D-dimensional anharmonic well $V^{(\mathrm{AHO})} \sim \vec{r}^2 + [\vec{r}^2]^2$. The proof combines a change of variables with Fourier transformation and throws a new light on the non-Hermitian Hamiltonian $H^{(\mathrm{BB})}$. It relates its parameter $J = D/2 + \ell - 1$ to the dimension D and angular momentum ℓ of its AHO partner. In the opposite direction it enables us to reinterpret certain AHO bound states as Fourier images of the exact BB bound states. These new AHO solutions could become represented by certain elementary integrals (cf also [3] in this respect).

One may feel dissatisfied by an incompleteness of the picture which prefers even dimensions and, hence, cannot be applied in some $D \gg 1$ models of nuclear physics [4]. In addition, in quantum chemistry [5] and atomic physics [6] a free variability of the dimension

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D would be welcome. Another reason for our interest in a possible generalization of the model (1) was given by an alternative version of the BG transformation [2]. It starts from a less usual, volcano-shaped D-dimensional model $V(\vec{r}) \sim \omega^2\vec{r}^2 - [\vec{r}^2]^2$ after a partial-wave decomposition and a complex shift of coordinates:

$$\begin{aligned} &H^{(\mathrm{BG})} = -\partial_t^2 + \frac{L(L+1)}{x^2} + \omega^2 x^2 - x^4 \qquad x = x(t) = t - \mathrm{i}\varepsilon \\ &L = \ell + (D-3)/2 \end{aligned} \tag{2}$$

with domain extended to the whole real line, $t \in \mathbb{R}$. Using a very similar transformation BG proved its isospectrality with a certain Hermitian double-well oscillator in one dimension. They arrived at a quadruple scheme:

$$\begin{array}{ccc} \boxed{\begin{array}{c}\text{non-Hermitian}\\ -x^4 \text{ BB model (1)}\end{array}} & \xleftrightarrow{x\to \mathrm{i}x} & \boxed{\begin{array}{c}\text{1-dimensional}\\ +x^4 \text{ double well}\end{array}} \\ \updownarrow \text{ BG transformation} & & \updownarrow \text{ BG transformation} \\ \boxed{\begin{array}{c}D\text{-dimensional}\\ +x^4 \text{ AHO well}\end{array}} & \xleftrightarrow{x\to \mathrm{i}(t-\mathrm{i}\varepsilon)} & \boxed{\begin{array}{c}\text{non-Hermitian}\\ -x^4 \text{ `volcano' (2)}\end{array}} \end{array}$$

where the two columns are related by the elementary changes of variables. The upper-left corner is partially solvable. In what follows we intend to show that the *singular* Hamiltonians of the lower-right corner type may equally well exhibit the same quasi-exact [7] solvability.

2. Charged BB potentials

Equations (1) and (2) exhibit the puzzling $\mathcal{PT}$ symmetry [8]

$$H^{(\mathrm{BB})} = \mathcal{PT} H^{(\mathrm{BB})} \mathcal{PT} \qquad \mathcal{P}\psi(x) = \psi(-x) \qquad \mathcal{T}\psi(x) = \psi^*(x).$$

Many similar complex models with the real spectrum may be defined on the real line [9]. Their potentials are formed by a spatially symmetric real well and its purely imaginary antisymmetric complement. An interpretation of these models parallels the usual bound-state problem. Once we switch the imaginary force off, the set of solutions splits into the convergent and divergent parts. In equation (2) we identify the convergent part with the standard physical solutions $\psi_{(\mathrm{qo})}(r) \sim r^{\ell+(D-1)/2}$ while the unphysical, quasi-even components $\psi_{(\mathrm{qe})}(r) \sim r^{(3-D)/2-\ell}$ diverge. For a simpler illustration one may use the harmonic oscillator [10] or a few other solvable or quasi-exactly solvable examples [11].

2.1. $\mathcal{PT}$-symmetric regularization

Models which weaken their hermiticity to the mere $\mathcal{PT}$ symmetry may also be defined off the real line [8, 12], on a curve within certain asymptotic sectors, where the appropriate $L^2(\mathbb{R})$ boundary conditions are imposed. In our examples (1) and (2) above, these sectors are defined as $S_k = \{x \in \mathbb{C}; x \neq 0, |\arg(x) - \frac{1}{6}(2k-1)\pi| < \frac{1}{6}\pi\}$. We may choose a curve $x^{(-)}(t)$ joining S_1 and S_3 (and accepted in [2]) or $x^{(+)}(t)$ which ends in S_4 and S_6 in accord with the recommendation of [1] (cf figure 1). The 'signature' $\sigma = \pm 1$ of the curve $x^{(\sigma)}(t)$ appears in the asymptotics

$$x^{(\sigma)}(t) \sim (\cos\varphi \pm \mathrm{i}\sigma \sin\varphi)t \qquad t \to \mp\infty \qquad \varphi \in \left(0, \frac{\pi}{3}\right). \tag{3}$$

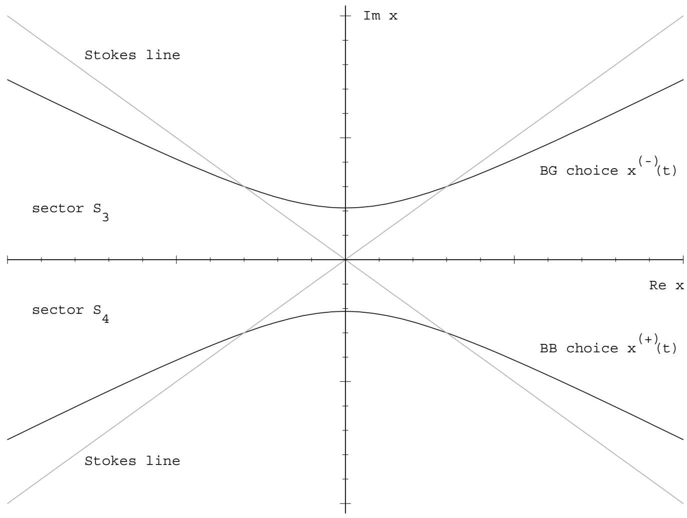

Figure 1. Integration paths for equation (3).

After any choice of σ we are going to study the Schrödinger equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{L(L+1)}{x^2}+\mathrm{i}\frac{d}{x}+\mathrm{i}cx+bx^2+\mathrm{i}ax^3-x^4\right]\psi(x)=E\psi(x) \tag{4}$$

which contains both the above bound-state problems (1) and (2) as its special cases. In the $\varphi \to 0$ straight-line extreme we have to use $2\varepsilon > |a|$ in $x^{(\sigma)}(t) = t - \mathrm{i}\sigma\varepsilon$. Whenever our curve $x^{(\sigma)}(t)$ opens downwards ($\sigma = +1$, [1]) or upwards ($\sigma = -1$, [2]) we have to cut the plane upwards or downwards, respectively. In this way our Schrödinger equation (4) with the integer or half-integer values of $L = \ell + (D-3)/2 = \frac{K}{2}$ remains perfectly regular along both our integration paths.

2.2. Recurrences

After a change of variables $x^{(\sigma)}(t) = -\mathrm{i}y(t)$ we get the new differential equation for $\psi(x) = \chi(\mathrm{i}x)$ which contains, formally, no imaginary units:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\frac{L(L+1)}{y^2}-\frac{d}{y}+cy-by^2-ay^3-y^4\right]\chi(y)=E\chi(y). \tag{5}$$

We shall search for its solutions by a power-series ansatz. Assuming that such a series terminates we arrive at a virtually unique formula

$$\chi(y)=\exp(\sigma\tfrac{1}{3}y^3+\tfrac{1}{2}Ty^2+Sy)\sum_{n=0}^{N}h_n y^{n-L}. \tag{6}$$

It combines both the respective y^{-L} and y^{L+1} quasi-even and quasi-odd components at threshold. Equation (5) determines

$$T = a/2\sigma \qquad S=(b-T^2)/2\sigma \qquad c\equiv c(N) = -2TS-\sigma\cdot(2N-2L+2)$$

i.e., the asymptotically correct values of our auxiliary parameters as well as the unique termination-compatible coupling c. The resulting ansatz (6) represents the desired normalizable solutions if and only if its coefficients comply with the recurrences

$$h_{n+1}A_n+h_nB_n+h_{n-1}C_n+h_{n-2}D_n=0 \qquad n=0,1,\ldots,N+1. \tag{7}$$

Their coefficients

$$A_n = (n+1)(n-2L) \qquad B_n = S(2n-2L) - d$$
$$C_n = S^2 + T(2n-2L-1) - E \qquad D_n = 2\sigma(n-N-2)$$

are all elementary.

3. Termination conditions

Our recurrences can be interpreted as an overdeterminate linear algebraic set:

$$\left(\begin{array}{ccccc} B_0 & A_0 & & & \\ \hline C_1 & B_1 & A_1 & & \\ D_2 & C_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_{N-1} & A_{N-1} \\ & & D_N & C_N & B_N \\ \hline & & & D_{N+1} & C_{N+1} \end{array} \right) \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix} = 0 \qquad (8)$$

of $N+2$ equations for $N+1$ unknown coefficients h_n. It is worth repeating that, by construction, our Schrödinger equation itself was regular along the curve $x^{(\sigma)}(t)$. Even at the non-vanishing $d \neq 0$ and $L \neq 0$, its two independent solutions remain equally admissible in the vicinity of the origin. After all, this was the main formal reason why the BB complex oscillator remained solvable by the ansatz of the form (6). At $L = 0$ it combined the quasi-even behaviour $\psi_{(\mathrm{qe})}(r) \sim \mathcal{O}(r^0)$ with its quasi-odd parallel $\psi_{(\mathrm{qo})}(r) \sim \mathcal{O}(r^1)$. Here we intend to proceed in the same manner at $L \neq 0$ and $d \neq 0$.

In a purely numerical setting it is important that the $(N+2) \times (N+1)$-dimensional non-square matrix equation (8) has two main diagonals:

$$B_n = S(2n-K) - d \qquad C_n = S^2 + T(2n-K-1) - E \qquad K = 2L.$$

This makes our linear set most easily solved as a coupled pair of the two square-matrix problems marked by the thick- and thin-line delimiters [13]. In such an arrangement, the quantities $d = d(E)$ and $E = E(d)$ play the role of two mutually coupled eigenvalues.

3.1. *Closed formulae for the energies*

As long as the upper diagonal $A_n = (n+1)(n-K)$ vanishes at $n = K$, there emerges an important asymmetry between our two eigenvalues d and E. Whenever one keeps just a few lowest partial waves, we may say that the integer $K = 2L$ remains small, at least in comparison with N, which can, or should, be large or at least arbitrary.

This is a fundamental observation leading to a thorough simplification of the construction. At the lowest possible $K = 0$ it returns us immediately to the BB proposal. Their choice of $d = 0$ in equation (1) (with $J = N+1$) gave them the highly welcome possibility of omitting the whole first row from equation (8). The rest of this equation is the usual square-matrix diagonalization. All the exceptional quasi-exact eigenvalues E_j, $j = 1, 2, \ldots, N+1$, can be determined as roots of a polynomial of the $(N+1)$th degree [1].

A transition to the nonzero integers K is less obvious. In place of using the trivial $d = 0$

we propose to satisfy the $(K+1)$-dimensional sub-equation

$$\det \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ D_2 & C_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_{K-1} & A_{K-1} \\ & & D_K & C_K & B_K \end{pmatrix} = 0. \tag{9}$$

This can fix, say, the eligible electric charges d as functions of the other parameters. These functions may be used as a starting point of a facilitated solution of our original problem (4). Thus, once we return to the trivial choice of $K = 0$ we can, in the simplest test of our general recipe, *derive* the trivial charge which has been *postulated* in [1]:

$$d = 0 \qquad K = 0.$$

In the next, truly innovative $K = 1$ step we get the nonvanishing double root $d = d_{(\pm)}(E) = \pm\sqrt{E}$. Each choice of the sign specifies a different relation between the charge and the energy. Both these signs stay compatible with the more compact inverse recipe $E = E_K(d)$,

$$E = E_1(d) = d^2 \qquad K = 1.$$

It defines the energy as a function of the charge. The role of the numbering of the separate elements of our new, $K = 1$, multiplets of bound states is taken over by the admissible charges $d = d_j$. The energy E may be eliminated from all our algebraic equations. The values of the charges d_j remain the only unknown quantities.

At the subsequent integer $K = 2$ we get, with a bit of luck, a highly compact energy formula

$$E = E_2(d) = \frac{d^2}{4} + 2\frac{ST + \sigma N}{d} \qquad K = 2 \qquad \sigma = \pm 1.$$

The next step gives the two rules or roots $E_3(d) = F$ of the quadratic equation

$$9F^2 - 10d^2F + d^4 + 48dST + 48d\sigma N - 24d - 72S - 36T^2 - 0 \qquad K = 3.$$

All the $K \geqslant 3$ cases require the so-called Gröbner elimination which should (and easily can) be performed by a computer [5].

3.2. *Closed formulae for the eigencharges*

The (real) values of the acceptable charges d_j have to follow from our termination postulates. At a *variable* integer N let us now assume that the value of K is *fixed*. Then, after the insertion of the energy $E = E_K(d)$ we are left with the upward recurrences (8), a square-matrix problem:

$$\begin{pmatrix} C_1(d) & B_1(d) & A_1 & & \\ D_2 & C_2(d) & \ddots & \ddots & \\ & \ddots & \ddots & B_{N-1}(d) & A_{N-1} \\ & & D_N & C_N(d) & B_N(d) \\ & & & D_{N+1} & C_{N+1}(d) \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix} = 0$$

of dimension $(N+1) \times (N+1)$. In a subsequent routine construction of the coefficients h_n of our wavefunctions the $K = 0$ results of [1] may be recalled and used as an illustration or guide.

Let us contemplate the first nontrivial option $K = 1$ for definitness. Insertion of the available formulae gives the explicit secular determinant

$$\det \begin{pmatrix} S^2-d^2 & S-d & 0 & & & \\ D_2(N) & S^2+2T-d^2 & 3S-d & 3 & & \\ & D_3(N) & S^2+4T-d^2 & \ddots & \ddots & \\ & & \ddots & \ddots & & \\ & & & D_{N+1}(N) & S^2+2NT-d^2 & \end{pmatrix} = 0 \qquad (10)$$

with the σ dependence in $D_j(N) = 2\sigma(n - N - 2)$. It exhibits certain symmetries. For example, the change of the signature $\sigma \to -\sigma$ may be compensated by the simultaneous sign change of $S \to -S$ and $d \to -d$. In addition, we see that an exceptional root $d = S$ exists at any $N \geqslant 0$. Its inspection reveals its physical acceptability. For example, at $N = 1$ this root is correct provided only that $S = -\sigma/2T$.

All the other roots of equation (10) become manifestly N-dependent. If needed, the boundaries of the domain of their reality may be determined numerically, in a complete parallel to the $K = 0$ study [1]. At $N = 1$ the three values of d_j follow from the cubic equation

$$-d^3 - Sd^2 + (S^2+2T)d + 2\sigma + S(S^2+2T) = 0 \qquad N = 1.$$

Their S and T dependence may be determined by Cardano formulae. In order to prove that the domain of their reality is non-empty let us only quote

$$d_{1,2,3,}(S,T) = (-5.303\,953\,910, -3.103\,253\,421, 5.407\,207\,331) \qquad (S,T) = (3,10).$$

The consequences of the growth of the dimension are well illustrated by the next two explicit polynomial secular equations generated at $K = \sigma = 1$:

$$\begin{aligned} & d^5 + Sd^4 - (2S^2+6T)d^3 - (6+2S^3+6ST)d^2 + (6S^2T+4S+8T^2+S^4)d \\ & \qquad +10S^2+16T+6S^3T+8ST^2+S^5 = 0 \qquad N = 2 \\ & -d^7 - Sd^6 + (12T+3S^2)d^5 + (12+12ST+3S^3)d^4 \\ & \qquad -(16S+44T^2+3S^4+24S^2T)d^3 \\ & \qquad -(40S^2+88T+44ST^2+3S^5+24S^3T)d^2 \\ & \qquad +(12+16S^3+S^6+44S^2T^2+64ST+48T^3+12S^4T)d \\ & \qquad +152S^2T+28S^4+84S+144T^2+S^7+44S^3T^2 \\ & \qquad +48ST^3+12S^5T = 0 \qquad N = 3. \end{aligned}$$

To this list one could add the next $N = 4$ item containing 53 terms, etc. This would complement the similar $K = 0$ formulae displayed in detail in [1] and show that the practical solution of equation (10) is more and more numerical with increasing N.

4. Discussion

An overall insight into the structure of spectra may be based on a quantitative study of a few simplified special cases. *A posteriori*, such an approach may be endorsed by a drastic reduction of secular polynomials. One may factorize many of them by purely analytic means and get more results in transparent form.

4.1. An extreme of simplicity: $S = T = K = 0$

In [1] the numerical analysis excluded the point $S = T = 0$ where, empirically, the $\mathcal{PT}$ symmetry becomes spontaneously broken. Some of the energies coalesce there and/or move off the real line in pairs. Our understanding of the $K = S = T = 0$ spectrum itself is incomplete although, for any N, the secular equation itself is quite transparent:

$$\det \begin{pmatrix} -E & 0 & 1\cdot 2 & & & \\ 2N\sigma & -E & 0 & 2\cdot 3 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 6\sigma & -E & 0 & (N-1)N \\ & & & 4\sigma & -E & 0 \\ & & & & 2\sigma & -E \end{pmatrix} = 0.$$

A σ independence of our simplified secular equations and their polynomiality in $x = E^3 \neq 0$ can be proved in an easy exercise for all N. Thus, at $N = 0$, $N = 1$ and $N = 2$ we get just the single real $E = 0$. The next two equations:

$$E^4 - 96E = 0 \qquad N = 3 \qquad -E^5 + 336E^2 = 0 \qquad N = 4$$

lead to the single nonzero real root $E \sim 4.48$ at $N = 3$ and $E \sim 6.95$ at $N = 4$, respectively, etc. The roots remain analytic up to $N = 13$. In the latter extreme we get the quadruplet of the real and nonzero energies $E \sim 9.381$, 17.768, 26.487 and 35.535.

4.2. A few $K = 1$ examples

After one moves to the negative signature $\sigma = -1$, only certain signs change in the secular polynomials at $S = T = 0$. They remain solvable up to $N = 6$. In detail, their first few samples

$$-d^3 + 2\sigma = 0 \qquad N = 1$$

offer the single real root $d \approx 1.26 \cdot \sigma$ at $N = 1$. One double zero and one real nonzero root $d \approx 1.71 \cdot \sigma$ result from the next equation:

$$d^5 - 6\sigma d^2 = 0 \qquad N = 2.$$

One simple zero and one positive and one negative root ($d_1 \approx 2.35 \cdot \sigma, d_2 \approx -0.975 \cdot \sigma$) follow from our third example:

$$-d^7 + 12\sigma d^4 + 12d = 0 \qquad N = 3.$$

Finally, our last polynomial sample equation

$$d^9 - 20\sigma d^6 - 76d^3 + 512\sigma = 0 \qquad N = 4$$

with solution $\sigma \cdot d_{1,2,3} \approx (2.82, 1.55, -1.83)$ illustrates the existence of the three different nonzero real roots.

4.3. The last analytic solution: $N = K + 1 = 3$

The choice of $N \leqslant K$ is extremely formal as it does not make any use of the rule $A_K = 0$. In this sense, the first reasonable $K = 2$ illustration has to use $N = 3$. After we fix $\sigma = +1$ for brevity, we get the secular polynomial of twelfth degree in d but, fortunately, the abbreviation $x = d^3 \neq 0$ reduces it to the solvable quartic equation

$$x^4 + 331\,776 - 96x^3 + 384x^2 + 18\,432x = 0 \qquad \sigma = 1$$
$$K = 2 \qquad N = 3.$$

It possesses the two real and positive roots

$$3x_1 = 24 \qquad x_2 = 24 + 16\sqrt[3]{9+\sqrt{17}} + \frac{64}{\sqrt[3]{9+\sqrt{17}}} \approx 88.872\,941\,16.$$

These roots lead to the real and positive charges $d_{1,2} \approx (2.88, 4.46)$ and to the $K = 2$ energies $E_2(d_j)$ as prescribed above. The parallel problem with $\sigma = -1$ leads to the full quadruplet of the negative real roots

$$x_{1,2,3,4} \approx (-199.78, -72.00, -14.65, -1.57) \qquad \sigma = -1$$

and to the eigencharges with the same signs, $d_{1,2,3,4} \approx (-5.84, -4.16, -2.45, -1.16)$.

We may conclude that at a fixed K (i.e., for a specific partial wave ℓ) we get, in general, a multiplet of states connected by some broken lines in the two-dimensional charge–energy plane. Only in the simplest $K = 0$ special case does this become the BB straight line $d = \text{constant} = 0$.

5. Summary and outlook

We have shown that the multiplets of exact bound-state solutions of the charged quartic oscillator can be constructed in closed form at any integer degree N and dimension D. Under certain relationship between the couplings and energies, arbitrarily large multiplets of bound states with real energies proved obtainable from a single and finite-dimensional secular equation.

The BB one-dimensional example reappears here as the simplest special case with the unique and, incidentally, vanishing Coulombic charge d. In our approach its construction finds one of its most natural interpretations. We have found that the generalized and charged multiplets lie along certain curves (e.g., parabolas) in the energy–charge space. In this sense, one finds a closer parallel to many other models of practical interest where the closed quasi-exact solutions are also numbered by the couplings. Many Hermitian models belong to this category, be it some popular non-polynomial example [14] or one of the historically first models of the quasi-exact type [15].

The existence of the new partially solvable model (4) opens up several new mathematical questions. For example, its interpretation from the modern Lie-algebraic point of view [16] is still missing. Moreover, the real mathematical meaning of its overall $\mathcal{PT}$ symmetry itself is not yet satisfactorily understood. It is only clear that the partially solvable status of our model is definitely due to its non-Hermitian character and, in effect, to the vanishing of the coefficient $A_K = 0$.

We may summarize that the first problem formulated in our introductory section is satisfactorily settled. In principle, the multiplet solvability occurs in all dimensions, i.e., not only for the s-waves in three dimensions and for the p-waves in one dimension at $K = 0$. In a systematic manner, the $K = 1$ case covers the s-waves in four dimensions and p-waves in two dimensions, we have to use $K = 2$ for s-waves in five dimensions and for p-waves in three dimensions, and we encounter the triple possibility of $(\ell, D) = (0, 6)$, $(1, 4)$ and $(2, 2)$ at $K = 3$, etc.

Our second initial motivation concerned the possible tractability of strong singularities of the centrifugal and Coulombic type. We have shown how they find a natural framework within the BG regularization scheme. In this way we achieved a satisfactory symmetry in the picture given at the end of section 1. Still, in the future, we intend to pay more attention to its 'hidden' $p \leftrightarrow x$ symmetry. At present it is not at all obvious how one could try to move beyond its known Fourier transformation [2] or quasi-harmonic-oscillator [17] manifestations.

In such a context, the centrifugal-like singularities (connected to the not yet explored changes of variables) do not seem to have had their last word yet.

Acknowledgments

My thanks belong to Rajkumar Roychoudhury (ISI, Calcutta) and to Francesco Cannata (INFN, Bologna) who insisted that the Bender–Boettcher model deserves a deeper study. Partial support by the GA AS CR grant Nr A 1048004 is acknowledged.

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